EQUILIBRIUM CONFIGURATION OF BLACK HOLES AND THE INVERSE SCATTERING METHOD*

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Abstract

The inverse scattering method is applied to the investigation of the equilibrium configuration of black holes. A study of the boundary problem corresponding to this configuration shows that any axially symmetric, stationary solution of the Einstein equations with disconnected event horizon must belong to the class of Belinskii-Zakharov solutions. Relationships between the angular momenta and angular velocities of black holes are derived.

1 Introduction

In the present paper, we study solutions to the Einstein equations in a vacuum that are stationary, axially symmetric, and asymptotically flat. The main purpose is to elaborate a description in which two classes of ideas, black hole theory and the theory of completely integrable equations, are unified.

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In black hole physics, the well-known result claims that "black holes have no hair". In other words, each black hole is uniquely determined by its mass and angular momentum and is nothing but the Kerr solution. Many authors contributed to this result and we do not have room to give the exhaustive references. For our purpose, only the final stage of the proof is important, namely, Carter's classification of all axially symmetric solutions having connected horizon [1]. Carter demonstrated that these solutions must solve the boundary problem for a system of elliptic nonlinear equations, the Kerr solution being one of the possible solution to this boundary problem. The uniqueness of this solution was proven in [2].

Can one do without requiring the connectedness of the horizon? In other words, do solutions exists that correspond to an equilibrium configuration of black holes? To the best of the author knowledge, these questions still have no exhaustive answers (though some results can be found in [3, 4]). Also, these questions can be formulated in terms of the boundary problem from [1] because, there, the connectedness of the horizon played no role. This boundary problem is formulated in the following section.

It is well known that the Einstein equations with two commuting symmetries belong to a wide class of systems that can be integrated by the methods of the inverse scattering problem. This was shown in [5] by Belinskii and Zakharov, who also investigated the axially symmetric case for which the $2N$-soliton solution was constructed [6]. This solution was interpreted as the solution corresponding to the N Kerr-NUT black holes. As we show in the present paper, the $2N$-soliton solution of Belinskii-Zakharov indeed contains all possible solutions (if any exist) corresponding to an equilibrium configuration of rotating black holes. Therefore, we reduce the question of the existence of solutions with disconnected horizon to the investigation of some subclass of the soliton solutions. This subclass is parameterized by the distances between black holes, the angular momenta, and the masses of the black holes. However, in general case, solutions from this have a conical singularity on the symmetry axis, which hinders the existence of solutions with disconnected horizon. It is still unclear whether it possible to choose the parameters in a way that removes this conical singularity. Most likely, the answer is negative [3, 4].

The presence of the conical singularity does not make these solutions physically meaningless. On the contrary, the conical singularity itself has the physical sense of the interaction force between the black holes. This interpretation was proposed by Weyl; for details see [7].

2 Boundary conditions

In this section, we present some basic facts about the stationary axially symmetric solutions to the Einstein equations and formulate the boundary problem corresponding to an equilibrium configuration for black holes. The results of [1] are crucial for us and we refer the reader there for details.

Recall that Lorentz manifold is stationary and axially symmetric if it possesses two commuting one-parameter isometry groups that are isomorphic to R and $SO(2)$ respectively. In other words, there exist two Killing vector fields k^a and m^a, which commute with each other, such that the vector k^a is time-like and the vector m^a is space-like. For the stationary and axially symmetric vacuum (or electrovacuum) space-time one can always choose the coordinate system in which the metric acquires the following form:

$$ds^2 = -Vdt^2 + 2Wdtd\phi + Xd\phi^2 + \frac{X}{\rho^2}e^{\beta}(d\rho^2 + dz^2).$$

Here the metric coefficients depend only on ρ and z, and $k^a\partial/\partial x^a = \partial/\partial t$, $m^a\partial/\partial x^a = \partial/\partial\phi$. Then V, W, X, ρ have a geometric meaning,

$$V = -k^a k_a,\ X = m^a m_a,\ W = k^a m_a, \tag{2.1a}$$

$$VX + W^2 = \rho^2 = -\rho_{ab}\rho^{ab}\ (\rho_{ab} = 2k_{[a}m_{b]}). \tag{2.1b}$$

The multitudes of the Killing vectors are normalized as follows: $V \to 1$ at infinity and $\frac{X^{,a}X_{,a}}{4X} \to 1$ on the symmetry axis. The coordinates ϕ, ρ and z become cylindrical (Weyl) coordinates.

The set of points with $\rho = 0, X = 0$ is the symmetry axis, while the set of points with $\rho = 0, X > 0$ is the event horizon. Let the event horizon have N connected components and l^a be an isotropic vector that is orthogonal to the event horizon. Then, in each connected component of the horizon, we can choose such a normalization of l^a that

$$l^a = k^a + \Omega_i m^a,$$

where Ω_i is some constant whose physical meaning is the angular velocity of the black hole.Let $z_1, z_2, \ldots, z_{2N}$ be the points of intersection of the horizon and the symmetry axis enumerated in increasing order.

We pass to a new coordinate system,

$$\rho^2 = (\lambda^2 - m_i^2)(1-\mu^2), m_i = \frac{z_{2i} - z_{2i-1}}{2},$$

$$z - \frac{z_{2i} + z_{2i-1}}{2} = \lambda\mu.$$

In this coordinate system, the necessary conditions of regularity of the symmetry axis and the horizon are formulated as follows:

$$\begin{aligned} X(\lambda,\mu) &= (1-\mu^2)\hat{X}(\lambda,\mu) \\ W^\dagger(\lambda,\mu) &= (\lambda^2 - m_i^2)(1-\mu^2)\hat{W}(\lambda,\mu) \\ V^\dagger(\lambda,\mu) &= (\lambda^2 - m_i^2)\hat{V}(\lambda,\mu) \end{aligned} \tag{2.2}$$

Here $\hat{X}, \hat{W}, \hat{V}$ are smooth functions nowhere equal to zero and

$$V^\dagger = -l^a l_a,\ W^\dagger = l^a m_a.$$

From (2.2) one can easily obtain the boundary conditions for the first group of Einstein equations,

$$d * \rho dg g^{-1} = 0,\ g = \begin{pmatrix} -V & W \\ W & X \end{pmatrix}, \tag{2.3}$$

where $*$ is the Hodge operator $*d\rho = dz, *dz = -d\rho$. Indeed using (2.2), one can easily prove that

$$\rho g_{,\rho} g^{-1} = \begin{pmatrix} 0 & O(1) \\ 0 & 2 \end{pmatrix},\ \rho \to 0,\ z \in \Gamma, \tag{2.4a}$$

$$\hat{\Omega}_i \rho g_{,\rho} g^{-1} \hat{\Omega}_i^{-1} = \begin{pmatrix} 2 & 0 \\ O(1) & 0 \end{pmatrix},\ \rho \to 0\ z \in I_i,\ \hat{\Omega}_i = \begin{pmatrix} 1 & \Omega_i \\ 0 & 1 \end{pmatrix}. \tag{2.4b}$$

$$\rho g_{,z} g^{-1} = O(1),\ \rho \to 0,\ z \in R. \tag{2.4c}$$

Here Γ is the symmetry axis consisting of $N+1$ connected components,

$$\Gamma = \bigcup \Gamma_j = R \setminus \bigcup I_i, \quad j = 1, \ldots, N+1, \quad i = 1, \ldots, N, \quad I_i = (z_{2i-1}, z_{2i}).$$

The symbol $O(1)$ denotes a uniformly bounded function on corresponding interval. It follows from (2.2) that (2.4c) tends to zero almost everywhere except the points z_k; however for our purposes, a uniform boundness suffices. The function $g(z,\rho)$ is taken to be smooth at all points except the points $(z_k, 0)$.

As we demonstrated below, Eqs. (2.4) completely determine the solution to Eqs. (2.3). At the same time Ω_i and z_i are the independent parameters of the boundary problem; these parameters can be choose arbitrary. The behavior of g at infinity is discussed at the end of this section.

An alternative approach, in which the main parameters are angular momenta rather than angular velocities, exists [7, 8]. To show this let us introduce the Ernst potentials

$$\rho * dgg^{-1} = \begin{pmatrix} -dY^{12} & d\tilde{Y} \\ -dY & dY^{21} \end{pmatrix}, \quad d(Y^{21} - Y^{12}) = 2dz. \tag{2.5}$$

Here Y is the Ernst potential that is determined by the space-like Killing vector field, while $\tilde{Y}$ is the Ernst potential determined by the time-like Killing vector field. Then system (2.3) can be rewritten in equivalent form,

$$d(\frac{\rho * dX}{X}) - \frac{\rho}{X^2} * dY \wedge dY = 0,$$

$$d(\frac{\rho * dY}{X}) + \frac{\rho}{X^2} * dX \wedge dY = 0, \tag{2.6a}$$

$$d\Omega = -\frac{\rho * dY}{X^2}, \quad W = \Omega X. \tag{2.6b}$$

From (2.4a) or (2.2) we can see that

$$\rho\partial_\rho \ln X \rightarrow 2,\ \rho \rightarrow 0,\ z \in \Gamma;\ Y|_{\Gamma_i} = c_i, \tag{2.7}$$

where c_i are some constants that are independent parameters. In [3, 7, 8], it was proved that (2.6) has a unique solution satisfying (2.7) and some condition at infinity for all z_i and c_i. In fact, the latter condition is equivalent to the asymptotic flatness of the metric. Note that (2.4b) follows from (2.6b) provided that Ω_i is defined as follows:

$$\Omega|_{I_i} = -\Omega_i, \tag{2.8}$$

Thus, Ω_i is a function of z_i and c_i.

The constants c_i unambiguously determine the angular momentum of all black holes. Indeed, let us define the angular momenta of a black hole using the Komar form:

$$L_i = \frac{1}{16\pi}\int_{S_i} m^{a;b} dS_{ab},$$

where S_i is a two-surface surrounding the black hole. Choosing S_i to be the surface of revolution of the curve C_i connecting the components of the axis Γ_{i+1} and Γ_i, we obtain

$$L_i = \frac{1}{8}\int_{C_i} \frac{1}{\rho} * (XdW - WdX) = \frac{1}{8}\int_{C_i} dY = \frac{1}{8}(c_{i+1} - c_i). \tag{2.9}$$

The mass of the black hole can be defined as the following integral [1]:

$$M_i = -\frac{1}{8\pi}\int_{S_i} k^{a;b} dS_{ab}.$$

Proceeding as for finding the angular momentum, we obtain that

$$M_i = \frac{1}{4}\int_{C_i} \frac{1}{\rho} * (XdV + WdW) = -\frac{1}{4}\int_{C_i} dY^{12}.$$

Contracting the contour C_i to the horizon and using (2.4b), we find that

$$M_i = \frac{1}{4}\int_{I_i} (2 + \Omega_i Y_{,z}) dz = m_i + 2\Omega_i L_i, \tag{2.10}$$

where $m_i = (z_{2j} - z_{2j-1})/2$.

At infinity, we impose the following conditions:

$$W = \rho^2 O(\frac{1}{r^3}), \quad X = \rho^2(1 + O(\frac{1}{r})). \tag{2.11}$$

where $r = \sqrt{\rho^2 + z^2}$. Asymptotic formulas (2.11) are assumed to be differentiable at least twice. Formulas (2.11) mean that the metric tensor g tends to the Minkowski tensor in cylindrical coordinates. From (2.11) we obtain

$$g_{,z}g^{-1} = \begin{pmatrix} O(1/r^2) & O(1/r^4) \\ \rho^2 O(1/r^4) & O(1/r^2) \end{pmatrix}, \quad V = 1 + O(1/r), \tag{2.12a}$$

$$\rho g_{,\rho} g^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} O(1/r) & O(1/r^3) \\ \rho^2 O(1/r^3) & O(1/r) \end{pmatrix}. \tag{2.12b}$$

When determining the angular velocities from (2.8), we should normalize Ω in accordance with (2.11), i.e. $\Omega = O(1/r^3)$.

The second group of Einstein equations allows one to determine the coefficient e^β from the matrix g. Using (2.4) or (2.7), one can show that $\partial_z \beta = 0$ for$\rho = 0$ and $z \in \Gamma$, i.e. $\beta|_{\Gamma_i} = b_i$, where b_i are some constant. The conical singularity on the symmetry axis is absent iff $b_i = 0$. However, b_i cannot be treated as independent parameters since they are functions of z_k and c_k. In the preset paper, we restrict ourselves to the study of boundary problem (2.4), (2.12) and do not discuss the properties of b_i.

3 Auxiliary linear problem

System of equations (2.3) is the compatibility condition for the following pair of matrix linear differential equations [5, 6]:

$$D_1\psi = \frac{\rho^2 g_{,z} g^{-1} - \omega\rho g_{,\rho} g^{-1}}{\omega^2 + \rho^2}\psi, \quad D_2\psi = \frac{\rho^2 g_{,\rho} g^{-1} + \omega\rho g_{,z} g^{-1}}{\omega^2 + \rho^2}\psi. \tag{3.1}$$

Here D_1 and D_2 are the commuting differential operators:

$$D_1 = \partial_z - \frac{2\omega^2}{\omega^2 + \rho^2}\partial_\omega, \quad D_2 = \partial_\rho + \frac{2\omega\rho}{\omega^2 + \rho^2}\partial_\omega,$$

and ω is a complex parameter that does not depend on the coordinates. We also use the $U - V$ pair representation in which ω is a dependent parameter. To be more precise, let ω be a root of the equation

$$\omega^2 - 2\omega(k - z) - \rho^2 = 0, \tag{3.2}$$

where k, in turn, is independent spectral parameter. Using (3.2), one can easily check that

$$\partial_z \omega = -\frac{2\omega^2}{\omega^2 + \rho^2}, \quad \partial_\rho \omega = \frac{2\omega\rho}{\omega^2 + \rho^2}. \tag{3.3}$$

Passing from $\psi(\omega)$ to $\psi'(k) = \psi(\omega(k))$, we obtain from (3.1) that

$$\partial_z \psi(k) = A(z, \rho, k)\psi(k), \quad A = \frac{\rho^2 g_{,z} g^{-1} - \omega\rho g_{,\rho} g^{-1}}{\omega^2 + \rho^2}, \tag{3.4a}$$

$$\partial_\rho \psi(k) = B(z,\rho,k)\psi(k), \quad B = \frac{\rho^2 g_{,\rho} g^{-1} + \omega\rho g_{,z} g^{-1}}{\omega^2+\rho^2} \qquad (3.4b)$$

Hereafter, we omit the prime for brevity. It is worth mentioning that Eqs. (3.4) are equivalent to Eqs. (3.1) only if we take ω to be the multivalued function in (3.4). Fixing the branch of the root in (3.4), we find the solution to system (3.1) only in the analyticity domain of $\omega(k)$.

In the present paper, we follow the general scheme for investigating integrable equations [9].

Since Eq.(3.2) is invariant with respect to the transformation $\omega \to -\rho^2/\omega$, we can fix the branch of the multivalued function $\omega(k)$, stipulating that the inequality $|\omega| > \rho$ holds. Then, from (3.2), we obtain

$$\omega \to 2(k-z),\ \rho \to 0;\ \omega \to 2(k-z),\ z \to \infty;\ \omega \to 2(k-z),\ k \to \infty \quad (3.5)$$

After choosing the branch of the root, we can introduce the monodromy matrix $T(z,y)$, which, by definition, is a solution to (3.4a) such that $T(y,y) = I$. Note that for $\rho \to 0$,

$$A(z,\rho,k) \to \frac{1}{2}\frac{1}{z-k}\begin{pmatrix} 0 & \partial_z \tilde{Y} \\ 0 & 2 \end{pmatrix}, \quad z \in \Gamma,$$

$$A(z,\rho,k) \to \frac{1}{2}\frac{1}{z-k}\hat{\Omega}_i^{-1}\begin{pmatrix} 2 & 0 \\ -\partial_z Y & 0 \end{pmatrix}\hat{\Omega}_i, \quad z \in I_i.$$

Here we took into account (3.5) and boundary condition (2.4). Hence Eq. (3.4a) can be easily integrated at $\rho = 0$. As a result, the explicit formulas for the monodromy matrix are

$$T(z,y) = \begin{pmatrix} 1 & -\frac{\tilde{Y}(z)-\tilde{Y}(y)}{2(k-y)} \\ 0 & \frac{k-z}{k-y} \end{pmatrix}, \quad z,y \in \Gamma_k, \qquad (3.7)$$

where Γ_k is the connected component of the symmetry axis and

$$T(z,y) = \hat{\Omega}_i^{-1}\begin{pmatrix} \frac{k-z}{k-y} & 0 \\ \frac{Y(z)-Y(y)}{2(k-y)} & 1 \end{pmatrix}\hat{\Omega}_i, \quad z,y \in I_i. \qquad (3.8)$$

Let $e(z,\rho,k)$ be the solution to (3.4) with the Minkowski tensor in cylindrical coordinates ($V = 1, W = 0, X = \rho^2$),

$$e(z,k) = \begin{pmatrix} 1 & 0 \\ 0 & \omega(z,k) \end{pmatrix},\ \partial_z e = A_0 e,\ A_0 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2\omega}{\omega^2+\rho^2} \end{pmatrix}. \qquad (3.9)$$

Let us define the Jost functions and reduced monodromy matrix,

$$\Psi^{\pm}(z,k) = \lim_{y\to\pm\infty} T(z,y)e(y), \tag{3,10}$$

$$T(k) = \lim_{y\to-\infty, z\to+\infty} e^{-1}(z)T(z,y)e(y). \tag{3.11}$$

We do not reproduce the explicit dependence on ρ. The functions $\Psi^{\pm}$ satisfy the following integral equations:

$$\Psi^{-}(z) = e(z) + \int_{-\infty}^{z} e(z)e^{-1}(x)A'(x)\Psi^{-}(x)dx, \tag{3.12}$$

$$\Psi^{+}(z) = e(z) - \int_{z}^{\infty} e(z)e^{-1}(x)A'(x)\Psi^{+}(x)dx, \tag{3.13}$$

where

$$A'(z) = A(z) - A_0(z) = \frac{\rho^2 g_{,z}g^{-1}}{\omega^2+\rho^2} - \frac{\omega}{\omega^2+\rho^2}\left(\rho g_{,\rho}g^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\right). \tag{3.14}$$

Limit (3.11) exists at least for $|\mathrm{Im}k| > \rho$. Further, using (3.4b), one can show that $T(k)$ does not depend on ρ. The basic property of the monodromy matrix reads

$$T(z,y) = T(z,z_{2N})T(z_{2N},z_{2N-1})\dots T(z_1,y), \quad z\in\Gamma_{N+1},\ y\in\Gamma_1, \tag{3.15}$$

where Γ_{N+1} and Γ_1 are the extreme components of the symmetry axis. Then, from (3.15), (3.11), (3.8), (3.7) and (3.5) we get that

$$T(k) = \hat{D}_{N+1}\prod_{j=1,\dots,N} T_j\hat{D}_j. \tag{3.16}$$

Here

$$T_j = \begin{pmatrix} 1-\frac{2M_j}{k-z_{2j-1}} & -4M_j\Omega_j \\ \frac{2L_j}{(k-z_{2j})(k-z_{2j-1})} & 1+\frac{2M_j}{k-z_{2j}} \end{pmatrix}, \quad \hat{D}_j = \begin{pmatrix} 1 & -D_j \\ 0 & 1 \end{pmatrix},$$

where the constant D_j are defined as follows:

$$D_{N+1} = (\tilde{Y}(\infty) - \tilde{Y}(z_{2N})),\ D_j = (\tilde{Y}(z_{2j-1}) - \tilde{Y}(z_{2j-2})),$$

$$D_1 = (\tilde{Y}(z_1) - \tilde{Y}(-\infty))$$

In (3.16), we took into account identities (2.9) and (2.10). Notice also that $\det T(k) = 1$.

The cut of $\omega(k, z, \rho)$ is the segment that connect the points $z + i\rho$ and $z - i\rho$. Therefore, $\Psi^{\pm}(k)$ are analytic functions in k as $|\mathrm{Im}k| > \rho$ and

$$\Psi^{-}(k) = \Psi^{+}(k)T(k). \tag{3.17}$$

The function $\Psi^{+}(k)$ ($\Psi^{-}(k)$) can be analytically continued in the domains $\mathrm{Re}k < z$ ($\mathrm{Re}k > z$). Using (3.5), we can see that for $k \to \infty$,

$$\Psi^{\pm}(k)e^{-1}(k) \to I \tag{3.18}$$

Substituting $k = z + (\omega^2 - \rho^2)/2\omega$ for k, we pass from $\Psi^{\pm}(k)$ to $\Psi^{\pm}(\omega)$. Then the function $\Psi^{\pm}(\omega)$ become solutions to Eqs. (3.1) in the domain $|\omega| > \rho$, while $\Psi^{+}(\omega)$ is analytic in ω as $\mathrm{Re}\omega < 0, |\omega| > \rho$ and $\Psi^{-}(\omega)$ is analytic as $\mathrm{Re}\omega > 0, |\omega| > \rho$. Furthermore, it follows from (3.18) that

$$\Psi^{\pm}(\omega)e^{-1}(\omega) \to I \tag{3.19}$$

as $\omega \to \infty$. Though $\Psi^{\pm}(\omega)$ are determined for $|\omega| < \rho$ as well, they do not satisfy system (3.1) in this domain. Therefore, our next aim is to continue $\Psi^{\pm}(\omega)$ into the domain $|\omega| < \rho$ in a manner that preserves Eqs.(3.1).

Let $\omega_1(k), \omega_2(k)$ be the roots of Eq. (3.2) and let $\mathrm{Re}\omega_1(k) < 0$ and $\mathrm{Re}\omega_2(k) > 0$. Then the cuts of $\omega_{1,2}(k)$ are half-lines going from points $z + i\rho$, $z - i\rho$ to infinity in a direction that is perpendicular to the real axis. Hence, the functions $\omega_{1,2}(k)$ are analytic for $|\mathrm{Im}k| < \rho$. Note that $\omega_1(k) = \omega(k)$ for $\mathrm{Re}k < z$ and $\omega_2(k) = \omega(k)$ for $\mathrm{Re}k > z$, whence the functions $\Psi^{\pm}(k)$ are continued analytically into the strip $|\mathrm{Im}k| < \rho$. Further, let $\Psi_1^{-}(k)$ and $\Psi_2^{+}(k)$ are the solutions to Eqs. (3.12) (with ω_1 substituted for ω) and (3.13) (with ω_2 substituted for ω), respectively .

For $|\mathrm{Im}k| < \rho$, the functions $\Psi^{+}(k)$ and $\Psi_1^{-}(k)$ ($\Psi^{-}(k)$ and $\Psi_2^{+}(k)$) are the solutions of the same differential equation (3.4a). Hence,

$$\Psi_1^{-}(k) = \Psi^{+}(k)T_1(k), \quad \Psi^{-}(k) = \Psi_2^{+}(k)T_2(k), \tag{3.20}$$

where the matrices $T_{1,2}(k)$ do not depend on z. Since the solutions to the integral equations (3.12) and (3.13) automatically satisfy (3.4b) (this follows from boundary conditions (2.12) and the fact that $e(z, \rho, k)$ is a common solution of Eq.(3.4) with the Minkowski tensor), we conclude that $T_{1,2}(k)$ do not depends on ρ as well. Moreover, at $\rho \to \infty$,

$$\omega_1(k) \to -\rho + (k-z) + O(\frac{1}{\rho}), \;\; \omega_2(k) \to \rho + (k-z) + O(\frac{1}{\rho}).$$

Therefore, accounting for Eq.(2.12), we derive that

$$\lim_{\rho\to\infty} e_1^{-1}(k)\Psi_1^-(k) = \lim_{\rho\to\infty} e_1^{-1}(k)\Psi^+(k) = I,$$

$$\lim_{\rho\to\infty} e_2^{-1}(k)\Psi^-(k) = \lim_{\rho\to\infty} e_2^{-1}(k)\Psi_2^+(k) = I, \tag{3.21}$$

and, hence, $T_1(k) = T_2(k) = I$. Here $e_1(k) = e(\omega_1(k))$ and $e_2(k) = e(\omega_2(k))$. In other words, for $|\mathrm{Im}k| < \rho$, we obtain

$$\Psi_1^-(k) = \Psi^+(k), \;\; \Psi^-(k) = \Psi_2^+(k). \tag{3.22}$$

The function $\Psi_1^-(k)$ and $\Psi_2^+(k)$ continued analytically into the domains $\mathrm{Re}k > z$ and $\mathrm{Re}k < z$ respectively. Hence, the function $\Psi_1^-(\omega)$ is analytic for $|\omega| < \rho$, $\mathrm{Re}\omega < 0$, and the function $\Psi_2^+(\omega)$ is analytic for $|\omega| < \rho$, $\mathrm{Re}\omega > 0$. As these functions are the solutions to Eq.(3.1) in the domain $\mathrm{Re}\omega < 0$ (Ψ_1^-) or in the domain $\mathrm{Re}\omega > 0$ (Ψ_2^+), it follows from (3.22) that $\Psi^+(\omega)$ can be analytically continued into the half-plane $\mathrm{Re}\omega < 0$ and remains a solution of (3.1), and $\Psi^-(\omega)$ can be analytically continued into the half-plane $\mathrm{Re}\omega > 0$, and also remains a solution of (3.1).

System (3.1) is invariant w. r. t. the transformation

$$\Psi(z,\rho,\omega) \to g\tilde{\Psi}^{-1}(z,\rho,-\frac{\rho^2}{\omega}), \tag{3.23}$$

which is valid because the matrix g is symmetric (the tilde in (3.23) denotes transposition). Reduction (3.23) means that $\Psi^-(k)$ and $g[\tilde{\Psi}_1^-(k)]^{-1}$ are the solutions of the compatible pair of equations (3.4) with $\omega = \omega_1$. However, at $\rho \to \infty$,

$$e_2^{-1}(k) g e_1^{-1}(k) \to -I. \tag{3.24}$$

Then from (3.21) and (3.24) we have that $\Psi^-(k) = -g[\tilde{\Psi}_1^-(k)]^{-1}$,or, equivalently,

$$\Psi^-(\omega) = -g[\tilde{\Psi}^+(-\frac{\rho^2}{\omega})]^{-1}. \tag{3.25}$$

The functions $\Psi^+(\omega)$ and $\Psi^-(\omega)$ satisfy the compatible system of equations (3.1). Therefore, the combination $[\Psi^+(\omega)]^{-1}\Psi^-(\omega)$ depends only on $k = z + (\omega^2 - \rho^2)/2\omega$ $(D_1 k = D_2 k = 0)$. Then, by virtue of identity (3.17),

$$\Psi^-(\omega) = \Psi^+(\omega) T(k). \tag{3.26}$$

Equations (3.26) and (3.25) show that

$$T(k) = \tilde{T}(k). \tag{3.27}$$

The monodromy matrix, $T(k)$ depends on $3N+1$ parameters except z_k. We treat equality (3.27) as the system of $2N+1$ nonlinear algebraic equations for the constants D_j, L_j and Ω_j:

$$\sum_{j=1}^{N+1} D_j + \sum_{j=1}^{N} 4\Omega_j M_j = 0, \ \mathrm{Res}_{z_k} T_{12}(k) = \mathrm{Res}_{z_k} T_{21}(k). \tag{3.28}$$

Assume D_j can be excluded from (3.28); then the remaining N equations give us the connection between the angular velocities and angular momenta. For instance, for the case of a single black hole, it follows from (3.28) that

$$D_1 = D_2 = -2M_1\Omega_1, \ \ \Omega_1 = \frac{L_1}{2M_1^2(m_1 + M_1)}. \tag{3.29}$$

The expression for Ω_1 in (3.29) is known; it establishes the connection between the angular velocity and angular momentum of the Kerr black hole. The relationship is expressed by

$$\Omega_1 = \frac{a}{(M_1 + m_1)^2 + a^2}, \ a = \frac{L_1}{M_1}, \ M_1^2 = m_1^2 + a^2.$$

As in (3.29), equality (2.10) is also taken into account here. Then the monodromy matrix reads

$$T(k) = \begin{pmatrix} 1 - \frac{2M_1}{k - z_1} + \frac{2M_1(M_1 - m_1)}{(k-z_1)(k-z_2)} & \frac{2L_1}{(k-z_1)(k-z_2)} \\ \frac{2L_1}{(k-z_1)(k-z_2)} & 1 + \frac{2M_1}{k-z_2} + \frac{2M_1(M_1-m_1)}{(k-z_1)(k-z_2)} \end{pmatrix}$$

Let us summarize the results of this section. Let boundary problem (2.4), (2.12) have a solution. Then there exists a piecewize analytic matrix $\Psi(\omega)$

($\Psi(\omega) = \Psi^+(\omega)$, Re$\omega < 0$ and $\Psi(\omega) = \Psi^-(\omega)$, Re$\omega > 0$) that satisfies the compatible system of linear equations (3.1), the conjugation condition on the imaginary axis,

$$\Psi_-(\omega) = \Psi_+(\omega)T(k), \quad \mathrm{Re}\omega = 0 \tag{3.30}$$

and the normalization condition at infinity,

$$\Psi(\omega)\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\omega} \end{pmatrix} \to I, \ \omega \to \infty. \tag{3.31}$$

In (3.30) $\Psi_\pm(\omega) = \lim_{\epsilon\to 0}\Psi(\omega \mp \epsilon)$ ($\epsilon > 0$).

The only singularities of the matrix $T(k)$ as a function of ω are simple poles at the points $\omega_i^\pm = (z_i - z) \pm \sqrt{(z_i - z)^2 + \rho^2}$. Since the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} T(k) \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix}$$

is regular at $\omega = 0$ and tends to I as $\omega \to \infty$ (which follows from explicit form of $T(k)$ and from Eqs. (3.27), (3.28)), we conclude that

$$\Psi^\pm(\omega) = \left(I + \sum_{j=1}^{2N} \frac{A_j^\pm}{\omega - \omega_j^\pm}\right)\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \tag{3.32}$$

where $A_j^\pm$ do not depend on ω.

Formula (3.32) demonstrate that a rational-in-ω solution to the auxiliary linear problem corresponds to each solution with a disconnected horizon and, hence, each such solution should be contained in the Belinskii-Zakharov class of solutions [6].

Assume that for any Ω_i, z_i (or L_i, z_i) the system of nonlinear equations (3.28) has a unique solution. Then a unique symmetric matrix $T(k)$ corresponds to each solution of the boundary problem (2.3), (2.4), and (2.12). However, since the solution to the Riemann problem (3.30), (3.31) is unique and g is unambiguously reconstructed by $\Psi(\omega)$ (see 3.25), we conclude that if a solution to the problem (2.3), (2.4), (2.12) exists, then it is unique as well. In particular, for the case where only one black hole is present, we obtain a new proof of the uniqueness of the Kerr solution.

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